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Note

Economising Plasma Simulation by Total Neglect of the Displacement Current

The modeling of plasmas using particles is a mature field of research [1], but standard explicit methods are numerically stable only when the space and time steps are sufficiently small to resolve all space and time scales. The most serious constraints arise from light waves, the Langmuir mode and the Debye length, λ_D [2, 3]. Considerable effort has recently been made to overcome these restrictions by using implicit schemes [4, 5].

We draw attention here to the advantages of neglecting the displacement current in allowing larger space and time steps in numerical models, and so extending their application. This eliminates light waves and the Langmuir mode from the model, and since the electric field must no longer be calculated from the charge density (because the quasineutral approximation has been made) the Debye length does not enter the problem.

Earlier work [6, 7] has shown how use of the Darwin limit of Maxwell's equations (i.e., neglecting the transverse displacement current) can eliminate light waves from the model while retaining many plasma properties. Nielson and Lewis [7] regard the total neglect of the dislacement current as naive, but it is common enough in analytical work, it significantly reduces the cost of computation relative to the Darwin model, and we believe that there is a range of modeling projects for which it is appropriate. The expense of large plasma models means we must justify inclusion, rather than neglect, of any term. An intelligent guess at the condition for importance of the longitudinal part of the displacement current is provided by the warm plasma model for ion acoustic waves. Writing $dp/d\rho = a^2$ for each species, the dispersion equation is

$$\frac{\omega_{pe}^2}{\omega^2 - k^2 a_e^2} + \frac{\omega_{pi}^2}{\omega^2 - k^2 a_i^2} = 1$$

and the right-hand side comes from the displacement current, so that neglect of this changes the right-hand side to zero. The Langmuir mode is apparent in the electron term and for the ion acoustic mode this term may be approximated by $-(k\lambda_D)^{-2}$ using $\lambda_D = a_e/\omega_{pe}$. Then

$$\omega^2 = k^2 a_i^2 + \frac{m_e}{m_i} \frac{k^2 a_e^2}{1 + k^2 \lambda_D^2}$$

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and neglect of the displacement current deletes $(1 + k^2\lambda_D^2)^{-1}$ in the last term. It is therefore reasonable to suppose that our condition is $k\lambda_D \ll 1$, and more vaguely, scales of the order of the collisionless skin depth or larger are suggested.

When the displacement current is totally neglected the key equations are Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{1}$$

and its time derivative in the form

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial \mathbf{J}}{\partial t}. \tag{2}$$

Equation (2) replaces Poisson's equation; that it is necessary to find an alternative has been explained by Hewett and Nielson [6] and the value of $\nabla \cdot \mathbf{E}$ will not be needed. The condition for the neglect of the longitudinal part of the displacement current in Eq. (2) is $\partial^2\mathbf{E}/\partial t^2 \ll \omega_p^2 \mathbf{E}$.

It might appear that electrostatic phenomena would then be excluded, but the electrostatic approximation to Eq. (1) is simply $\mathbf{J} = 0$ which, for instance, allows acoustic waves.

Although the solution of Eq. (2) may be somewhat involved it will be worthwhile if the time step is sufficiently increased by the method. $\partial\mathbf{J}/\partial t$ must be expressed as a function of $\mathbf{E}$ using the plasma model, which can take a variety of forms. Our comments are based on the separation of $\partial\mathbf{J}/\partial t$ into two parts:

$$\partial\mathbf{J}/\partial t = (\partial\mathbf{J}/\partial t)_0 + \alpha(\mathbf{x}, t)\mathbf{E}, \tag{3}$$

where $(\partial\mathbf{J}/\partial t)_0$ is simply the value $\partial\mathbf{J}/\partial t$ would have if $\mathbf{E}$ were zero. For fluid components $(\partial\mathbf{J}/\partial t)_0$ can be obtained from stored fluid variables, as Nielson and Lewis [7] have pointed out. But for components where this is not possible, for example, particle or waterbag representations, $\mathbf{J}$ must be obtained from the model, and $(\partial\mathbf{J}/\partial t)_0$ obtained by running the model with zero $\mathbf{E}$ for one time step.

We will first examine a hybrid plasma model which includes both fluid and kinetic electrons. This allows us to satisfy Eq. (1) by using it to find the fluid electron current, and quasineutrality is used to set the electron fluid density.

The term $\alpha(\mathbf{x}, t)\mathbf{E}$ in Eq. (3) represents the dependence of $\partial\mathbf{J}/\partial t$ on $\mathbf{E}$, which is surely linear. We can apply very basic plasma theory [8] which obtains $\partial\mathbf{J}/\partial t$ from the first moments of the Vlasov equations for each species, and it is easily seen that in this case $\alpha(\mathbf{x}, t)$ is simply $\varepsilon_0\omega_p^2$, which must be supplied from the model. A common approximation is neglect of the time derivative of the current in Ohm's Law, when the electric field is then given algebraically. In some astrophysical problems, however, such as magnetospheric neutral sheets, and in some laboratory plasmas, the scale lengths approach the collisionless skin depth so we must retain this term. This case, $\alpha(\mathbf{x}, t) = \varepsilon_0\omega_p^2$, where ω_p is a function of space and time via the density, will serve to clarify the nature of the equation to be solved for **E**. Substitution into Eq. (2) gives

$$\nabla \times (\nabla \times \mathbf{E}) + \frac{\omega_p^2}{c^2}\mathbf{E} = -\mu_0 \left(\frac{\partial \mathbf{J}}{\partial t}\right)_0. \tag{4}$$

In two dimensions considerable simplification of this coupled pair of partial differential equations is possible. If $\mathbf{E} = (E_x, E_y, 0)$, and we write p for the z-component of $\nabla \times \mathbf{E}$, then multiplying Eq. (2) by $\lambda_c^2 = c^2/\omega_p^2$ and taking the curl gives

$$\nabla \cdot (\lambda_c^2 \nabla p) - p = \mu_0 \hat{\mathbf{z}} \cdot \nabla \times \left[\lambda_c^2 \left(\frac{\partial \mathbf{J}}{\partial t}\right)_0\right]. \tag{5}$$

This clarifies the boundary conditions which can be imposed on the electric field. Since the differential operator is elliptic we could apply Dirichlet conditions, but it may be more useful in physical problems to use Neumann conditions, since the normal derivative can be specified by E_x and E_y using Eq. (4):

$$\frac{\partial p}{\partial y} + E_x/\lambda_c^2 = -\mu_0 \left(\frac{\partial J_x}{\partial t}\right)_0, \tag{6}$$

$$-\frac{\partial p}{\partial x} + E_y/\lambda_c^2 = -\mu_0 \left(\frac{\partial J_y}{\partial t}\right)_0. \tag{7}$$

Equations (6) and (7) are then used to find the electric field once p has been found. The differenced form of Eq. (5) will be symmetric and diagonally dominant, and could be efficiently solved using, for example, the ICCG method [9]. Equation (5) provides a clear demonstration of the economy obtained relative to the Darwin model as it is of the same form as one of the equations whose solution is required by Hewett and Nielson [6]. While Eq. (5) is the only expensive equation in our field solver theirs involves three elliptic equations, including one which requires iteration.

For the case of plasma models without fluid components, particle or waterbag models, for example, it requires more effort to self-consistently satisfy Eq. (2) (and hence Eq. (1), which is no longer used) at each time step, and some iteration is necessary. This is because $\partial \mathbf{J}/\partial t$ at one mesh point may depend on not only the **E** value at that point but also on the **E** values at surrounding points. In a particle model, for instance, interpolated **E** values must be used to push the particles. The dominant effect is proportional to the difference between **E** at a mesh point and the mean of the **E** values at surrounding mesh points, i.e., proportional to $\nabla^2 \mathbf{E}$, so that its importance depends on the scale length for **E** relative to the mesh spacing. If the field is smooth relative to the mesh, iteration should converge rapidly enough. Since $(\omega_p^2/c^2)\mathbf{E}$ will still be the leading component of $\partial \mathbf{J}/\partial t$ we suggest the following iteration scheme for solving Eq. (2):

$$\frac{\omega_p^2}{c^2}\mathbf{E}^{n+1} = \nabla \times (\nabla \times \mathbf{E}^n) - \left(\frac{\partial \mathbf{J}}{\partial t}\right)^n + \frac{\omega_p^2}{c^2}\mathbf{E}^n, \tag{8}$$

where $(\partial \mathbf{J}/\partial t)^n$ is found from the model by time-stepping the particles using $\mathbf{E}^n$ for the electric field. It would be expected that the limitation on the time step relates to the time taken for a particle to cross a cell, but explicit particle models are similarly limited in relation to the calculation of acceleration of single particles.

The ideas described here have been used so far in only a restricted linear simulation, but Eqs. (2) and (3) are always linear. The linearity allows us to Fourier transform in one of the two space dimensions, so that Eq. (2) becomes a pair of one-dimensional differential equations. Equations (2) and (3) were used to obtain values of $\mathbf{E}$ and these were used to push the kinetic electrons, the ions and the magnetic field, but the velocity of the fluid electrons was found from Eq. (1). Another simplification, due to linearisation, was that α was the same at each time step.

The actual method of solving Eq. (2) for $\mathbf{E}$ was to notice that Fourier transformation in the x-direction allows the E_x component to be eliminated from Eq. (2).

If

$$\mathbf{E} = e^{ikx}(E_x(y), E_y(y), 0)$$

then E_y can be found from a second-order ordinary differential equation by the well-known tridiagonal matrix algorithm. The convenient and not unreasonable boundary condition $E_y = 0$ at both boundaries was successfully employed. E_x can then be found from an algebraic equation involving source terms and the first derivative of E_y.

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